# Technical Aspects

## Auction Settings

There are $n+1$ players, indexed by $i \in \{0,1,\ldots,n\}$ a non-participating auctioneer *(player 0)* and $n$ bidders. There is a single item for auction. Bidders have a common value $v$ for the item[5]. There is a set of potentially unbounded periods, indexed by $t \in \{0,1,2,3\ldots\}$[6]. Each period is characterized by a publicly-observable current leader $l_t \in \{0,1,2,3\ldots n\}$ with $l_o=0$. In each period $\boldsymbol{t}$, bidders simultaneously choose $x^i_t \in \{Bid, Not\ Bid\}$. If any of the bidders bid, one of these bids is randomly accepted[7]. In this case, the corresponding bidder becomes the leader for the next period and pays a non-refundable cost $\boldsymbol{c}$. If none of the players bids, the game *ends at period t* and the current leader receives the object[8].

In addition to the bid costs, the winner of the auction must pay a *bid amount*. The bid amount starts at 0 and weakly rises by the *bidding increment* $k \in R^+$ in each period, so that the bid amount for the good at time $t$ is $tk$ (note that the bid amount and time are deterministically linked). Therefore, at the end of the game, the auctioneer payoff consists of the final bid amount $(tk)$ along with the total collected bid costs $(tc)$.

I assume that players are risk neutral and do not discount future consumption. Assume that $c < v - k$ so that there is the potential for bidding in equilibrium. To match the empirical game, I assume that the current leader of the auction cannot place a bid[9]. I often refer to the net value of the good in period $t$ as $v\text{-}tk$. Consequently refer to auctions with $k > 0$ as $(k)$ declining-value auctions and auctions with $k= 0$ as constant-value auctions.

Model the game in discrete time in order to capture important qualitative characteristics that cannot be modeled in continuous time (such as the ability to bid or not bid in each individual period regardless of past choices). However, the discreteness of the game requires an additional technical assumption for declining-value auctions that $mod(v-c, k)=0$. If this condition is not satisfied, the game unravels and there is no equilibria in which play continues past the first period[10].

For simplicity, will focus on Markov-Perfect Equilibria. [11,12] Bidder $i$'s Markov strategy set consists of a bidding probability for every period given that she is a non-leader $\{p^i_0, p^i_1, p^i_2, ..., p^i_t\}$ with $p^i_t \in [0,1]$. Commonly make statements about the discrete hazard function, $h(t,l_t) \equiv P[x^i_t = \text{Not Bid}]$ *for all* $i \neq l_t | \text{Reaching period } t \text{ with leader } l_t]$, which is a function that maps each state (a period and potential leader) to the probability that the game ends, conditional on the state being reached.

Note that $h(0,0)=\Pi_i (1-p^i_0)$ and $h(t,l_t)=\Pi_i \neq l\ (1-p^i_t)$.

Finally, for expositional purposes, I define two measures of profit for the auctioneer throughout the game. To understand these concepts, note that the bidder $i$ at period $t-1$ is paying the auctioneer a bid cost c in exchange for a probability of $h(t,i)$ of winning the net value of the good $(v-tk)$ at time t. In other words, the auctioneer is selling bidder i a stochastic good with an expected value of $h(t,i)(v-tk)$ for a price $c$ at time $t$. Therefore, define the instantaneous profit of the auctioneer at time $t$ with leader $l_t$ as $\pi|(t,|lt) = c - h(t,l_t)(v-tk)$ and the instantaneous percent markup as: $\mu(t,lt)= \left(\frac{\pi(t,l\ )}{h(t,l\ )(v-tk)}\right) \bullet 100.$

## Footnotes

[5]I assume that the item is worthy v<v to the auctioneer. The case in which bidders have independent private values vi~G for the item is discussed in the online appendix. As might be expected, as the distribution of private values approaches the degenerate case of one common value, the empirical predictions converge to that of the common values case.

[6]It is important to note that t does not represent a countdown timer or clock time. Rather, it represents a "bidding stage," which advances when any player makes a bid.

[7]In current real-life implementations of this auction, two simultaneous bids would be counted in (essentially) random order. Modeling this extension is difficult, especially with a large number of players, as it allows the time period to potentially "jump." Hinnosaar (2013) theoretically analyzes this change (combined with other changes to model) and finds a multiplicity of very complicated equilibria. In the online appendix, I show that the predictions of my model become much more complicated, but remain qualitatively similar when this assumption is changed in isolation.

[8]Note that, unlike the real-world implementation, there is no "timer" that counts down to the end of each bidding round in this model. As discussed in the online appendix, the addition of a timer complicates the model without producing any substantial insights; any equilibrium in a model with a timer can be converted into equilibrium without a timer that has the same expected outcomes and payoffs for each player.

[9]This assumption has no effect on the bidding equilibrium in Proposition 2 below, as the leader will not bid in equilibrium even when given the option. However, the assumption does dramatically simplify the exact form of other potential equilibria, as I discuss in the online appendix.

[10] discuss this issue in detail in the online appendix. While the equilibrium in Proposition 2 no longer exists if the condition does not hold, the strategies constitute a contemporaneous-perfect equilibrium for an extremely small E (on the order of hundredths of pennies) given the observed empirical parameters.

[11]A Markov-Perfect Equilibrium (Maskin and Tirole 2001) is a refinement of subgame perfection in which players are restricted to condition strategies only on payoff-relevant outcomes. In penny auctions, this removes seemingly-odd equilibria in which players coordinate bidding strategies depending on the identity of the current leader (i.e. player 10 bids if player 1 is the leader, while player 9 bids if player 2 is the leader).

[12]As I show in the online appendix, the statements for hazard rates all hold true when non-Markovian strategies are used.

# Equilibrium

While there are many hazard functions and strategy sets that can occur in equilibrium, I argue that it is appropriate to focus on a particular function and set (identified in Proposition 2) as these must occur in any state that is reached on the equilibrium path after period 1.
To begin the analysis, Proposition 1 notes the relatively obvious fact that no player will bid in equilibrium once the cost of a bid is greater than the net value of the good in the following period, leading the game to end with certainty in any history when this time period is reached.

Proposition 1 $F = vc/k - 1$ if $k > 0$
If $k > 0$, then in any Markov Perfect Equilibrium, the game never continues past period $F$.
That is, $h(t, l_t) = 1$ if $t > F$.
I refer to the set of periods that satisfy this condition as the...nal stageof the game. Note that there is no final stage of a constant-value auction, as the net value of the object does not fall and therefore this condition is never satisfied. With this constraint in mind, Proposition 2 establishes the existence of an equilibrium in which bidding occurs in each period $t \leq F$

Proposition 2 There exists a Markov Perfect Equilibrium in which players' strategies, the hazard rate, and auctioneer profits over time are described by

(A)
$$p_i^t = \begin{cases} 1 & \text{for } t=0 \\ 1-\sqrt[n-1]{\dfrac{c}{v-tk}} & \text{for } 0 < t \leq F \text{ for all } i \end{cases}$$

(B)
$$h(t, l_t) = \begin{cases} 0 & t=0 \\ \dfrac{c}{v-tk} & \text{for } 0 < t \leq F \text{ for all } l_t \end{cases}$$

(C)
$$\pi(t, l_t) = \{ \; 0 \text{ for any } t \text{ for all } l$$

In an equilibrium with this hazard function, players bid symmetrically such that the hazard rate in all histories after time $0$ and up to period $\frac{c}{v-tk}$. This hazard rate at time period $t$ causes the expected value from bidding (and the auctioneer profit) in all histories at period $t-1$ to be zero, leading players in these histories to be indifferent between bidding and not bidding.

This allows players in $t-1$ to use strictly mix ed behavioral strategies such that the hazard rate in all histories in period $\frac{c}{v-(t-1)k}$ which causes the players in period $t-2$ to be indifferent, and so on. Crucially, in a declining-value auction, there is no positive deviation to players in period $F$, who are indifferent given that players in period $F+1$ bid with zero probability, (which they must do by Proposition 1).

Note that, in the hazard function in Proposition 2, $h(0,0)=0$ is a (arbitrarily) assumption. This choice does not change any of the results in the paper, but simply implies that some bidding always occurs in equilibrium. This is the only choice in which the auctioneer expected revenue is $v$, which might be considered the natural outcome in a common-value auction[13].

For a constant-value auction, the strategies are equivalent to those in a symmetric discrete-time war-of-attrition (WOA) when $n$ players remain in the game. However, the hazard rate for the WOA is higher as play only continues if more than one player bids, whereas play in a penny auction continues if any player bids.[14]

Not surprisingly, there is a continuum of other equilibria in this model. In some of these equilibria, players (correctly) believe that some player will bid with very high probability in period 1 or 2, respectively, which leads them to strictly prefer to not bid in the previous period.[15] Consequently, the auction always ends at period 0 or period 1. Surprisingly, Proposition 3 notes that if we ever observe bidding past period 1, we must observe the hazard rates in Proposition 2 for all periods following period 1. If additionally all $n$ players meaningfully participate in the start of the auction (bid with some probability in the initial two periods), players must be following the individual strategies in Proposition 2 for all periods following period 1.

Proposition 3

For declining-value auctions $(k>0)$, in any Markov Perfect Equilibrium

(A) Any observed hazard rate $h(t,l_t)$ follows Proposition $2$ for $t>1$.

(B) Individual strategies pit follow Proposition 2 for $t>1$ if $p^i_o>O$ and $p^i_1>0$ for all $i$.

For constant-value auctions $(k=0)$, these statements are true when restricting to symmetric strategies.

To understand the intuition for statement (1) when $k>0$, consider some period $t$ with $1<t<F$ in which $h(t,l\ )\neq \frac{c}{v-tk}$ As a result of this hazard rate, player lt must strictly prefer either to bid or not bid in period $t-1$. If she prefers not to bid, then $(t,l\ )$ will not be reached in equilibrium (and will never be observed). Alternatively, if she prefers to bid, then it must be that $h(t-1,l\ -1)=0$ for any $l\ -1\neq l$ , leading all players other than i to strictly prefer to not bid in period t-2. Therefore, player $l$ cannot be a non-leader in period $t-1$ in equilibrium, so $(t,l\ )$ will not be observed in equilibrium.

Proposition 3 can also be interpreted as an "instantaneous zero-profit" condition on the equilibrium path. The expected hazard rate $\frac{c}{v-tk}$ leads to zero expected profits. If this condition is violated, players in $t-1$ or $t-2$ bid in a way that keeps the state on of the equilibrium path.

Statement (2) requires the additional constraint that each player bids with some probability when $t = 0$ and $t = 1$. The constraint excludes equilibria in which one player effectively leaves the game after period 0 (leading to $n-1$ players in the game) and in which some player is always the leader in a specific period (allowing her strategy for that period to be or-the-equilibrium path and therefore inconsequential). For intuition as to why strategies must be symmetric, consider the case in which players $i$ and $j$ choose strategies such that $pit \neq pjt$ *for some* $t > 1$ . Then, it must be that the players face different hazards as the leader in period

$t : h(t|l \;\; = i) \neq h(t|l \;\; = j)$ , leading one of these hazards to not equals $\frac{c}{v-tk}$ which leads to the issues discussed above.

Finally, note that the statements when $k = 0$ require the additional assumption of symmetric strategies.[16] Unlike in declining-value auctions, there is a somewhat complicated non-symmetric equilibrium in which three players alternatively cycle between bidding with relatively low probability and certainty. While players still expect to make zero profits from each bid over time, the hazard rate oscillates around $\frac{c}{v-tk}$ between periods. Choose not to focus on this type of equilibrium because this behavior requires heavy coordination among players and do not observe these oscillations empirically. Additionally, in the majority of my auction-level data and all of my bid-level data, $k > 0$.

## Footnotes

[13]Furthermore, if the auctioneer values the item at less than v, then he strictly prefers that bidding occurs in period 0, while the bidders are indifferent. If the auctioneer can select the equilibrium (or repeat the auction until some bidder bids in period 0), he would euectively select the particular equilibrium in Proposition 2.

[14]Another intuitive explanation for this difference is that, in equilibrium, the expected total costs (the bid costs of all players) spent in each period must equal the expected total benefit (the hazard rate times the value of the good). In the penny auction, only one player ever pays a bid cost at each period, whereas in the WOA, there is a chance that more than one player must pay the bid cost. Therefore, the benefit (determined by the hazard rate) must be higher in the WOA.

[15]There do not exist similar asymmetric equilibria in which the auction always ends in period 2 (or later). If this occurred, all non-leaders would strictly prefer to bid in period 1. Therefore, the auction would never end in period 1. But then all bidders in period 0 could never win the auction and would strictly prefer to not bid, causing the auction to never reach period 1.
"There are symmetric Markov equilibria that do not lead to the same hazard rates as those in Proposition 2. For example, consider the equilibrium in which all players always bid in odd (even) periods and never bid in even (odd) periods. In this equilibrium, the game always ends after period 0 (period 1).

[16]There are symmetric Markov equilibria that do not lead to the same hazard rates as those in Proposition 2. For example, consider the equilibrium in which all players always bid in odd (even) periods and never bid in even (odd) periods. In this equilibrium, the game always ends after period 0 (period 1).

# The Sunk Cost Fallasy

As I will show in Section 4, the predictions of zero pro...ts from the model above are strongly empirically violated. Therefore, in this section, I preemptively present an alternative model that better matches the patterns in the data. In this simple alternative model, players suffer from a sunk cost fallacy, in that they become more willing to bid as their bid costs rise, even though these costs are sunk. This model is a simplified and modified version of the sunk-cost model introduced in Eyster (2002), in which players desire to take present decisions (continuing to invest in a bad project) to justify their past decisions (investing in the project initially).

To capture sunk costs in the most parsimonious and portable way, I simply assume that each player's perception of the value of the good rises as she spends more money on bid costs, capturing an additional bene...t from justifying her sunk investment. Specifically, a player $i$ who has placed $s_i$ bids has sunk costs $s_i c$ and perceives the value of the good as $v + \Theta s_i c$ with $\Theta \geq 0$ defined as the sunk cost parameter. As this parameter rises, the player sunk costs cause her to bid with a higher likelyhood in the auction. If this parameter is zero, the model reverts to the standard model above.

Assume that the player is naïve about this sunk-cost effect, in the sense that she is unaware that her perception of value might change in the future and that she is unaware that other players do not necessarily share her value perception. Without the first type of naivety, players would be aware that they will bid too much in the future and consequently require a compensating premium to play the game initially, leading to zero profits for the auctioneer (and violating the empirical observations). Without the second type of naivety, each player would have very complicated higher-order beliefs, being personally unaware of her own future changes in value perception, but being aware of other players changing perceptions and being aware of other players'(correct) beliefs about her own changing perceptions. Furthermore, due to the mechanics of mixed strategy equilibria, each player (non-Markovian) bidding probability would largely be determined by the sunk costs of other players rather than her own sunk cost[17]. With this naivety assumption, a player simply plays the game as if the value of the good matches her perceived value, which includes a portion of her own sunk costs[18]

The sunk costs faced by a player at a specific time $t$ depend on the realizations of the players own mixing decisions, the mixing decisions of the other players, and the realization of the leader selection process. Define $s^t_i$ as the sunk bids placed by player $i$ at time $t$ for a particular realization of the game. Define $\vec{s}$ as the vector containing all of the player sunk bids and extend $p^t_i$ to be dependent on $s^t_i$ and $h(t,1_t)$ ant $\pi(t,1t)$ to be dependent on $\vec{s}$. Given this adjustment, Proposition 4 mirrors Proposition 2:

Proposition 4 *With sunk costs, there exists a Markov Perfect Equilibrium in which players' strategies, the hazard rate, and auctioneer profits over time are described by*

$$(A)\ p^t_i(s^t_i) = \begin{cases} 1 & \text{for } t = 0 \\ 1 - \sqrt[n-1]{\frac{c}{v - tk + \geq \Theta s^t_i c}} & \text{for } 0 < t \leq F \\ \text{for all } i & \\ 1 & \text{for } t > F \end{cases}$$

$$(B)\ h(t,l_t,\vec{s}) = \begin{cases} 0 & t = 0 \\ \prod i \neq l \boxtimes \sqrt[n-1]{\frac{c}{v - tk + \geq \Theta s^t_i c}} & \text{for } 0 < t \leq F \\ \text{for all } l_t & \\ 1 & \text{for } t > F \end{cases}$$

$$(C)\ \pi(t,l_t,\vec{s}) = \{\ c - h(t,l\ ,\vec{s})(v - tk) \quad \text{for any } t \text{ for all } l$$

If $\Theta = 0$, *(A), (B), and (C)* matches those in Proposition 2.

These formulas depend on the specific distribution of sunk costs across the players in each game. For expositional purposes, consider the simplifying assumption that $s_i c = \frac{1}{n} tc$ that is, sunk costs are distributed equally across players). In this case $h(t,l\ ,\vec{s})\ \frac{c}{v - tk + \geq \Theta s^t_i c}$. While this formula will likely not be satisfied in an individual realization of the game, it is helpful in understanding the comparative statics of the hazard rate and to provide a rough interpretation of the results when the individual distribution of sunk costs is unknown.

## Footnotes

[17]In a mixed strategy equilibrium, each player' probability of bidding in the following period is chosen to make the other players indifferent between bidding in the current period. A player is still affected by her own sunk costs as she will not bid if the current bid amount is above her own perceived value.

[18]More specifically, following other examples of naivety in the literature, an equilibrium requires that each player actions weakly maximizes her own payoff given the players (potentially mistaken) beliefs of other players' actions. In turn, the player' beliefs about other players' actions must constitute an equilibrium for the game that the player perceives she is playing.

# Summary

Propositions 2 and 4 predict a variety of comparative statics about the hazard rate of the auction and bidding behavior, both with and without a sunk cost fallacy.

If players do not super from a sunk cost fallacy, the hazard rate is $\frac{c}{v\text{-}tk}$ and players bid with probability $\sqrt[n\text{-}1]{\frac{c}{v\text{-}tk}}$ when $0 < t \leq F$, and the auctioneer profits remain constant at 0. A few comparative statics are of note. First, none of the parameters affect the auctioneer instantaneous profits, which remain at zero throughout the auction. Second, for constant-value auctions $(k = 0)$, the hazard rate and individual bidding probabilities remain constant throughout the auction. For declining-value auctions $(k > 0)$, individuals bid less in the auction as it proceeds (and the net value of the good is falling), leading to a higher hazard rate. This erect is strengthened as the bid increment $k$ rises. Third, as the number of players increases, each player's equilibrium bidding probability drops, but the hazard rate and profits stay constant.[19] Intuitively, the specific hazard rate in Proposition 2 can be interpreted as a zero profit condition, which must hold regardless of the number of the players. This is useful empirically, as cannot directly observe the number of players in the auction-level data. Finally, as the value of the good rises, individuals bid with higher probability and the hazard rate consequently decreases. As a result, auctions with higher values continue longer in expectation.

The final comparative static warrants a short digression. As the empirical data consist of many goods that take many values, the auctions are not predicted to share the same survival rates. This divergence creates a challenge in creating a visual representation of the predicted and empirical hazard rates. However, as I discuss in detail in the online appendix, this problem can be solved by using the concept of normalized time $t^\wedge = \frac{t}{v}$ . The basic intuition is that, given a constant bidding increment $k$, an auction with a good of value $v$ is approximately as likely to survive past time $t$ as an auction with a good of value $2v$ surviving past time $2t$, with the relationship approaching equality as the length of periods approaches zero.[20] That is, all auctions have approximately the same survival rates in normalized time. As a result, hazard rates in normalized time are approximately the same for these auctions, allowing auctions with different values $v$ to be compared. Note that the use of normalized time does not equalize survival rates across auctions with different bidding increments $k$, which consequently must be grouped into different visual representation[21].

When players super a sunk cost fallacy, the hazard rate is $\Pi_{i \neq l_t} \sqrt[n-1]{\frac{c}{v+\theta \dot{s}_i^t c - tk}}$ and players bid with probability $1-\sqrt[n-1]{\frac{c}{v+\theta \dot{s}_i^t c - tk}}$ when $0 < t \leq F$, and the auctioneer profits are $c - h(t, l_t \vec{s})(v - tk)$. There are a few important changes in the comparative statics from the standard model. For the hazard rate and bidding)probabTties, the erect is most easily seen for a constant-value auction (when the hazard is $\Pi_{i \neq l_t} \sqrt[n-1]{\frac{c}{v+\theta \dot{s}_i^t c}}$ . Rather than remaining constant over time, the hazard rate starts at the point predicted by the standard theory, but falls farther from this baseline as the auction continues. This occurs because individuals start with no sunk costs, but bid with higher probability as their personal sunk costs rise from paying for past accepted bids. This gradual deviation from the standard predictions also occurs in declining-value auctions, although it is possible that bidding probabilities rise due to the erect of the bid amount (which rises over time) overweighing the sunk-cost erect. This ambiguity does not occur when focusing on instantaneous profits (or profit margins).

## Footnotes

191n the model, the exact number of players in the auction is common knowledge. More realistically, the number of players could be drawn from a commonly-known distribution. In this case, players will bid such that expected auctioneer profits are still zero. However, when the specific realization of the number of players is low (high), the auctioneer will make negative (positive) profits.

20For example, the probability that a constant-value auction (k= 0) with bid cost c=l and value v=100 survives to time t=50 is (1-1/100)500~0.605, while the corresponding probability with value v=200 is (1-1/200)100~0.606. The comparable survival probabilities for these auctions given k=1 are 0.495 and 0.497.

21It is less clear how to construct a similar normalized time measure to compare auctions with different bidding increments. Most notably, an constant-value auction (with k=0) has a non-zero survival rate at every period, while the survival rate is always zero after the final stage of a declining-value auction (k>0).